Units, measurement, uncertainties and vectors

1.1 Measurement in physics

Physics is an experimental science in which measurements made must be expressed in units. In the international system of units used throughout this book, the SI system, there are seven fundamental units, which are defined in this section. All quantities are expressed in terms of these units directly, or as a combination of them.

The SI system

The SI system (short for Système International d'Unités) has seven **fundamental base units**. Since May 2019, new definitions have been given for all seven base units. The new definitions make use of seven physical constants whose numerical value is taken to be exact and fixed.

Unit	Symbol	Quantity	Natural constant involved
Second	s	Time	Δf_{CS}
Metre	m	Length	c
Kilogram	kg	Mass	h
Ampere	A	Electric current	e
Kelvin	K	Temperature	k_B
Mole	mol	Amount of substance	N_A
Candela	cd	Luminous intensity	K_{CD}

Thus the kilogram, for example, which used to be defined in terms of a certain quantity of a platinum–iridium alloy kept at the Bureau International des Poids et Mesures in France, is now defined in terms of the Planck constant, h.

1 The **second** (s). This is the unit of time. The second is that length of time so that the electromagnetic radiation emitted in a transition between the two hyperfine energy levels in the ground state of a caesium-133 atom has the exact frequency Δf_{CS} = 9 192 631 770 s^{-1}. Thus, $1\text{ s} = \dfrac{9\,192\,631\,770}{\Delta f_{CS}}$.

2 The **metre** (m). This is the unit of distance. The metre is that length so that the speed of light has the exact value $c = 299\,792\,458\text{ m s}^{-1}$ and so

$$1\text{m} = \frac{c}{299\,792\,458} \times \text{s} = \frac{c}{299\,792\,458} \times \frac{9\,192\,631\,770}{\Delta f_{CS}} = \frac{c}{\Delta f_{CS}} \times 3.066\,331\,899 \times 10^{1}.$$

3 The **kilogram** (kg). This is the unit of mass. The kg is that mass for which the Planck constant has the exact value $h = 6.626\,070\,15 \times 10^{-34}\text{ kg m}^2\text{ s}^{-1}$ and so

$$1\text{ kg} = \frac{h}{6.626\,070\,15\times 10^{-34}}\times\frac{\text{s}}{\text{m}^2} = \frac{h}{6.626\,070\,15\times 10^{-34}}\times\frac{\dfrac{9\,192\,631\,770}{\Delta f_{\text{CS}}}}{(\dfrac{c}{\Delta f_{\text{CS}}}\times 3.066\,331\,899\times 10^{1})^2}$$

$$= \frac{h\Delta f_{\text{CS}}}{c^2}\times 1.475\,514\,00\times 10^{40}$$

4 The **ampere** (A). This is the unit of electric current. 1 A is that current for which the electric charge transferred within 1 s is 1 Coulomb. The elementary charge is exactly $e = 1.602\,176\,634\times 10^{-19}$ C. Thus

$$1\text{ A} = \frac{1\text{ C}}{1\text{ s}} = \frac{e}{1.602\,176\,634\times 10^{-19}}\times\frac{\Delta f_{\text{CS}}}{9\,192\,631\,770} = e\Delta f_{\text{CS}}\times 6.789\,686\,817\times 10^{8}.$$

5 The **kelvin** (K). This is the unit of temperature. One kelvin is that temperature for which the product with the Boltzmann, $k_{\text{B}}\times 1\text{ K}$ is exactly equal to $1.380\,649\times 10^{-23}\text{ J} = 1.380\,649\times 10^{-23}\text{ kg m}^2\text{ s}^{-2}$. Hence

$$1\text{K} = \frac{1.380\,649\times 10^{-23}}{k_{\text{B}}}\times\frac{h\Delta f_{\text{CS}}}{c^2}\times 1.475\,514\,00\times 10^{40}\times\left(\frac{c}{\Delta f_{\text{CS}}}\times 3.066\,331\,899\times 10^{1}\right)^2\times\left(\frac{9\,192\,631\,770}{\Delta f_{\text{CS}}}\right)^{-2}$$

$$= \frac{h\Delta f_{\text{CS}}}{k_{\text{B}}}\times 2.266\,665\,265$$

6 The **mole** (mol). This is the unit of amount of substances. One mole of a substance contains as many particles as the Avogadro number which has the exact value $N_{\text{A}} = 6.022\,140\,760\times 10^{23}$. Thus

$$1\text{mol} = \frac{1}{N_{\text{A}}}\times 6.022\,140\,760\times 10^{23}.$$

7 The **candela** (cd). We will not deal with this unit at all and we will not define it.

You do not need to know the details of these definitions.

Physical quantities other than those above have units that are combinations of the seven fundamental units. They have **derived** units. For example, speed has units of distance over time, metres per second (i.e. m/s or, preferably, m s^{-1}). Acceleration has units of metres per second squared (i.e. m/s^2, which we write as m s^{-2}). Similarly, the unit of force is the newton (N). It equals the combination kg m s^{-2}. Energy, a very important quantity in physics, has the joule (J) as its unit. The joule is the combination N m and so equals (kg m s^{-2} m), or kg m^2 s^{-2}. The quantity power has units of energy per unit of time, and so is measured in J s^{-1}. This combination is called a watt (W). Thus:

$$1\text{W} = (1\text{J s}^{-1}) = \left(1\text{N m s}^{-1}\right) = \left(1\text{kgm s}^{-2}\text{m s}^{-1}\right) = 1\text{kgm}^2\text{ s}^{-3}$$

Pressure is measured in a unit called pascal (Pa) and equals $1\text{Pa} = 1\text{N m}^{-2} = 1\text{kg m s}^{-2}\text{ m}^{-2} = 1\text{kg m}^{-1}\text{ s}^{-2}$.

Working backwards, a quantity with units kg s^{-3} must be:
$\text{kg s}^{-3} = (\text{kgms}^{-2})\text{s}^{-1}\text{m}^{-1} = \text{N s}^{-1}\text{m}^{-1} = (\text{N m})\text{ s}^{-1}\text{m}^{-2} = \text{J s}^{-1}\text{m}^{-2} = \text{W m}^{-2}$ and so represents power per unit area i.e. intensity.

Metric multipliers

Small or large quantities can be expressed in terms of units that are related to the basic ones by powers of 10. Thus, a nanometre (nm) is 10^{-9} m, a microgram (μg) is 10^{-6} g = 10^{-9} kg, a gigaelectron volt (GeV) equals 10^{9} eV, etc. The most common prefixes are given in Table **1**.

Power	Prefix	Symbol	Power	Prefix	Symbol
10^{-18}	atto-	a	10^{1}	deka-	da
10^{-15}	femto-	f	10^{2}	hecto-	h
10^{-12}	pico-	p	10^{3}	kilo-	k
10^{-9}	nano-	n	10^{6}	mega-	M
10^{-6}	micro-	μ	10^{9}	giga-	G
10^{-3}	milli-	m	10^{12}	tera-	T
10^{-2}	centi-	c	10^{15}	peta-	P
10^{-1}	deci-	d	10^{18}	exa-	E

Table 1 Common prefixes in the SI system.

Orders of magnitude and estimates

Expressing a quantity as a plain power of 10, the exponent of 10 is what is called the **order of magnitude** of that quantity. Thus, the mass of the universe which is 10^{53} kg has an order of magnitude of 53; the mass of the Milky Way galaxy is 10^{41} kg and so has an order of magnitude of 41. The ratio of the two masses is then simply 10^{12}, an order of magnitude 12.

Tables **2**, **3** and **4** give examples of some interesting lengths, masses and times.

Lengths/ m	
distance to edge of observable universe	10^{26}
distance to the Andromeda galaxy	10^{22}
diameter of the Milky Way galaxy	10^{21}
distance to nearest star (other than the Sun)	10^{16}
diameter of the solar system	10^{13}
distance to the Sun	10^{11}
radius of the Earth	10^{7}
size of a cell	10^{-5}
size of a hydrogen atom	10^{-10}
size of an A = 50 nucleus	10^{-15}
size of a proton	10^{-15}
Planck length	10^{-35}

Table 2 Some interesting distances.

Masses/ kg	
the Universe	10^{53}
the Milky Way galaxy	10^{41}
the Sun	10^{30}
the Earth	10^{24}
Antonov AN-225 (fully loaded)	10^{6}
an apple	0.2
a raindrop	10^{-6}
a bacterium	10^{-15}
smallest virus	10^{-21}
a hydrogen atom	10^{-27}
an electron	10^{-30}

Table 3 Some interesting masses.

Times/ s	
age of the Universe	10^{17}
age of the Earth	10^{17}
time of travel by light to nearby star (other than the Sun)	10^{8}
one year	10^{7}
one day	10^{5}
period of a heartbeat	1
lifetime of a pion	10^{-8}

lifetime of the omega particle	10^{-10}
time of passage of light across a proton	10^{-24}

Table 4 Some interesting times.

Worked examples

1.1 Estimate how many grains of sand are required to fill the volume of the Earth. (This is a classic problem that goes back to Aristotle. The radius of the Earth is about 6 × 10^6 m.)

Answer

The volume of the Earth is:

$$\frac{4}{3}\pi R^3 \approx \frac{4}{3}\times 3\times\left(6\times 10^6\right)^3 \approx 8\times 10^{20} \approx 10^{21}\,\mathrm{m}^3$$

The diameter of a grain of sand varies of course, but we will take 1 mm as a fair estimate. The volume of a grain of sand is about $(1 \times 10^{-3})^3$ $\mathrm{m}^{3.}$

Then the number of grains of sand required to fill the Earth is:

$$\frac{10^{21}}{\left(1\times 10^{-3}\right)^3} \approx 10^{30}$$

1.2 Estimate the speed with which human hair grows.

Worked examples

I have my hair cut every two months and the barber cuts a length of about 2 cm. The speed is therefore:

$$\begin{aligned}\frac{2\times 10^{-2}}{2\times 30\times 24\times 60\times 60}\,\mathrm{ms}^{-1} &\approx \frac{10^{-2}}{3\times 2\times 36\times 10^4}\\ &\approx \frac{10^{-6}}{6\times 40} = \frac{10^{-6}}{240}\\ &\approx 4\times 10^{-9}\,\mathrm{m\,s}^{-1}\end{aligned}$$

1.3 Estimate how long the line would be if all the people on Earth were to hold hands in a straight line. Calculate how many times it would wrap around the Earth at the equator. (The radius of the Earth is about 6 × 10^6 m.)

Answer

Assume that each person has his or her hands stretched out to a distance of 1.5 m and that the population of Earth is 8 × 10^9 people.

Then the length of the line of people would be $8\times 10^9 \times 1.5\,\mathrm{m} \approx 10^{10}\,\mathrm{m}$.

The circumference of the Earth is $2\pi R \approx 6\times 6\times 10^6\,\mathrm{m} \approx 4\times 10^7\,\mathrm{m}$.

So the line would wrap $\frac{10^{10}}{4\times 10^7} \approx 250$ times around the equator.

1.4 Estimate how many apples it takes to have a combined mass equal to that of an ordinary family car.

Answer

Assume that an apple has a mass of 0.2 kg and a car has a mass of 1400 kg.

Then the number of apples is $\dfrac{1400}{0.2} = 7\times10^3$.

1.5 Estimate the time it takes light to arrive at Earth from the Sun. (The Earth–Sun distance is 1.5×10^{11} m.)

Answer

The time taken is $\dfrac{\text{distance}}{\text{speed}} = \dfrac{1.5\times10^{11}}{3\times10^8} \approx 0.5\times10^3 = 500\,\text{s} \approx 8\,\text{min}$

Significant figures

The **number** of digits used to express a number carries information about how precisely the number is known. A stopwatch reading of 3.2 s (two significant figures, s.f.) is less precise than a reading of 3.23 s (three s.f.).

How to find the number of significant figures in a number is illustrated in Table **5**.

Number	Number of s.f.	Reason	Scientific notation
504	3	in an integer all digits count (if last digit is not zero)	5.04×10^2
1200	2	zeros at the end of an integer do not count	1.2×10^3
200	1	zeros at the end of an integer do not count	2×10^2
0.000 305	3	zeros in front of a decimal do not count	3.05×10^{-4}
0.000 90	2	zeros at the end of a decimal count, those in front do not	9.0×10^{-4}

Table 5 Rules for significant figures.

Scientific notation means writing a number in the form $a \times 10^b$, where a is decimal such that $1 \le a < 10$ and b is a positive or negative integer. The number of significant figures in a is the number of significant figures in $a \times 10^b$.

In multiplication or division (or in raising a number to a power or taking a root), the result must have as many significant figures as the **least** precisely known number entering the calculation. So we have that:

$$\underset{2\text{ s.f.}}{23}\times\underset{3\text{ s.f.}}{578} = 13294 \approx \underbrace{1.3\times10^4}_{2\text{ s.f.}}$$

$$\frac{\underset{4\text{ s.f.}}{6.244}}{\underset{2\text{s.f.}}{1.2}} = 5.20333\ldots \approx \underbrace{5.2\times10^0}_{2\text{ s.f.}} = 5.2$$

$$\underset{3\text{ s.f.}}{12.3^3} = 1860.867\ldots \approx \underbrace{1.86\times10^3}_{3\text{ s.f.}}$$

$$\underbrace{\sqrt{58000}}_{2\text{ s.f.}} = 240.8318\ldots \approx \underbrace{2.4\times10^{2}}_{2\text{ s.f.}}$$

In adding and subtracting, the number of decimal digits in the answer must be equal to the least number of decimal places in the numbers added or subtracted. Thus:

$$\underset{2\text{d.p.}}{3.21} + \underset{1\text{d.p.}}{4.1} = 7.32 \approx \underset{1\text{d.p.}}{7.3}$$

$$\underbrace{12.367}_{3\text{ d.p.}} - \underset{2\text{ d.p.}}{3.15} = 9.217 \approx \underset{2\text{ d.p.}}{9.22}$$

Use the rules for rounding when writing values to the correct number of decimal places or significant figures. For example, the number 542.48 = 5.4248×10^2 rounded to 2, 3 and 4 s.f. becomes:

$5.4|248\times10^2 \approx 5.4\times10^2$ rounded to 2 s.f.

$5.42|48\times10^2 \approx 5.42\times10^2$ rounded to 3 s.f.

$5.424|8\times10^2 \approx 5.425\times10^2$ rounded to 4 s.f.

There is a special rule for rounding when the last digit to be dropped is 5 and it is followed only by zeros, or not followed by any other digit. This is the odd–even rounding rule. For example, consider the number 3.250 000 0… where the zeros continue indefinitely. How does this number round to 2 s.f.? Because the digit before the 5 is **even** we do not round up, so 3.250 000 0… becomes 3.2. But 3.350 000 0… rounds up to 3.4 because the digit before the 5 is **odd**. (There is no universal agreement about this rule.)

Nature of science

Early work on electricity and magnetism was hampered by the use of different systems of units in different parts of the world. Scientists realised they needed to have a common system of units in order to learn from each other's work and reproduce experimental results described by others. Following an international review of units that began in 1948, the SI system was introduced in 1960. At that time there were six base units. In 1971 the mole was added, bringing the number of base units to the seven in use today. A major review took place in 2019 with the result that the seven base units are now defined in terms of fixed fundamental constants.

As the instruments used to measure quantities have developed, the definitions of standard units have been refined to reflect the greater precision possible. Using the transition of the caesium-133 atom to measure time has meant that smaller intervals of time can be measured accurately. The SI system continues to evolve to meet the demands of scientists across the world. Increasing precision in measurement allows scientists to notice smaller differences between results, but there is always uncertainty in any experimental result. There are no 'exact' answers.

Test yourself

1 How long does light take to travel across a proton?
2 How many hydrogen atoms does it take to make up the mass of the Earth?
3 What is the age of the universe expressed in units of the Planck time?
4 How many heartbeats are there in the lifetime of a person (85 years)?
5 What is the mass of our galaxy in terms of a solar mass?
6 What is the diameter of our galaxy in terms of the astronomical unit, i.e. the distance between the Earth and the Sun (1 AU = 1.5×10^{11} m)?
7 The molar mass of water is 18 g mol^{-1}. How many molecules of water are there in a glass of water (mass of water 200 g)?

8 Assuming that the mass of a person is made up entirely of water, how many molecules are there in a human body (of mass 60 kg)?

9 Give an order-of-magnitude estimate of the density of a proton.

10 How long does light take to traverse the diameter of the solar system (10^{13} m)?

11 An electron volt (eV) is a unit of energy equal to 1.6×10^{-19} J. An electron has a kinetic energy of 2.5 eV.

a How many joules is that?

b What is the energy in eV of an electron that has an energy of 8.6×10^{-18} J?

12 What is the volume in cubic metres of a cube of side 2.8 cm?

13 What is the side in metres of a cube that has a volume of 588 cubic millimetres?

14 Give an order-of-magnitude estimate for the mass of:

a an apple

b this physics book

c a soccer ball.

15 A white dwarf star has a mass about that of the Sun and a radius about that of the Earth. Give an order-of-magnitude estimate of the density of a white dwarf.

16 A sports car accelerates from rest to 100 km per hour in 4.0 s. What fraction of the acceleration due to gravity is the car's acceleration?

17 Give an order-of-magnitude estimate for the number of electrons in your body.

18 Give an order-of-magnitude estimate for the ratio of the electric force between two electrons 1 m apart to the gravitational force between the electrons.

19 The frequency *f* of oscillation (a quantity with units of inverse seconds) of a mass *m* attached to a spring of spring constant *k* (a quantity with units of force per length) is related to *m* and *k*. By writing $f = cm^x k^y$ and matching units on both sides, show that $f = c\sqrt{\frac{k}{m}}$, where *c* is a dimensionless constant.

20 A block of mass 1.2 kg is raised a vertical distance of 5.55 m in 2.450 s. Calculate the power delivered. $\left(P = \frac{mgh}{t} \text{ and } g = 9.81\,\mathrm{m\,s^{-2}}\right)$

21 Find the kinetic energy $\left(E_K = \frac{1}{2}mv^2\right)$ of a block of mass 5.00 kg moving at a speed of 12. 5 m s^{-1}.

22 Without using a calculator, **estimate** the value of the following expressions. Then compare your estimate with the exact value found using a calculator.

a $\frac{243}{43}$

b 2.80×1.90

c $312 \times \frac{480}{160}$

d $\frac{8.99 \times 10^9 \times 7 \times 10^{-16} \times 7 \times 10^{-6}}{(8 \times 10^2)^2}$

e $\frac{6.6 \times 10^{-11} \times 6 \times 10^{24}}{(6.4 \times 10^6)^2}$

1.2 Systematic and random uncertainties

This section introduces the basic methods of dealing with experimental uncertainty. Physics is an experimental science and often the experimenter will perform an experiment to test the prediction of a given theory or measure the value of a physical quantity. No measurement will ever be completely accurate, however, and so the result of every experiment will be presented with an experimental uncertainty.

Types of uncertainty

There are two main types of uncertainty in a measurement. They can be grouped into **systematic** and **random**, although in many cases it is not possible to distinguish clearly between the two. We may say that random uncertainties are almost always the fault of the observer, whereas systematic uncertainties are due to both the observer and the instrument being used.

Systematic uncertainties

A **systematic uncertainty** makes all measurements either larger or smaller than the true value. If you use a metal ruler which was calibrated at $20^{\circ}C$ to measure length of a wooden box on a very hot day with temperature $40^{\circ}C$, your measurements will be too small because the metal ruler expanded in the hot weather (more than the wood). If you use an ammeter that shows a current of 0.1 A even before it is connected to a circuit, every measurement of current made with this ammeter will be larger than the true value of the current by 0.1 A. This is called a **zero error**.

You cannot eliminate systematic uncertainties by measuring the same quantity very many times and then taking an average. You have to check your instruments and re-evaluate your methods of taking measurements.

There will be a systematic uncertainty in measuring the volume of a liquid inside a graduated cylinder if the tube is not exactly vertical. The measured values will always be larger or smaller than the true value, depending on which side of the cylinder you look at (Figure **1a**). There will also be a systematic uncertainty if your line of sight is not normal to the scale (Figure **1b**). These are called parallax errors.

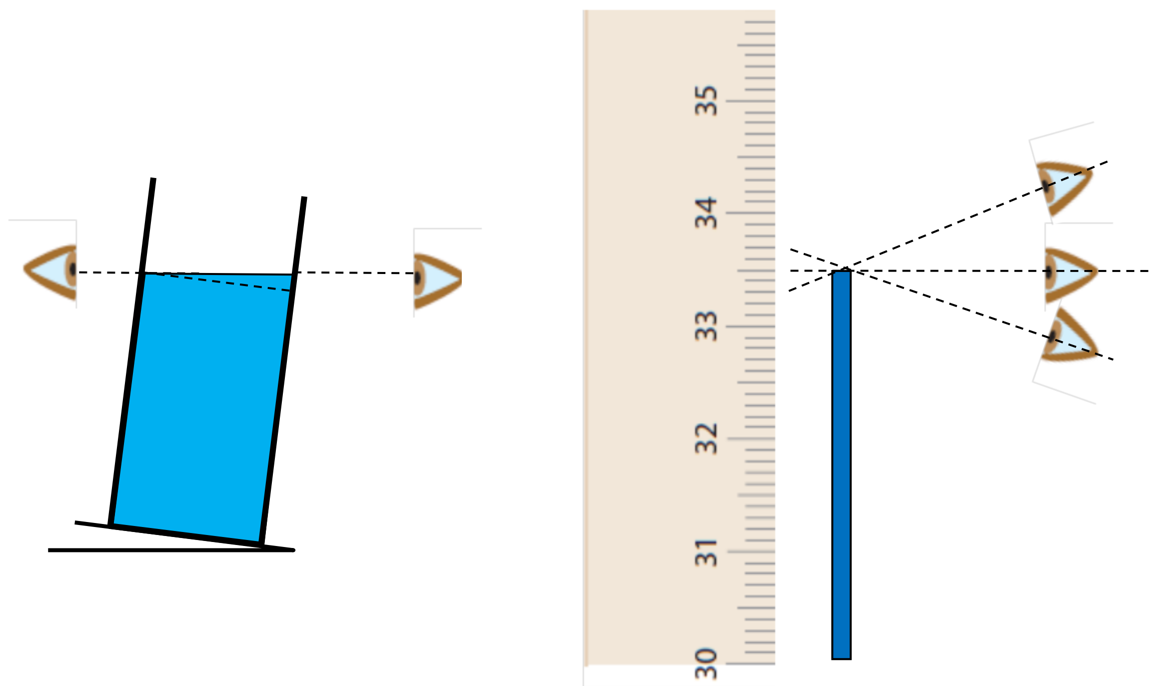


Figure 1 Parallax errors in measurement.

Another source of systematic uncertainty is the reaction time of a human in starting and stopping a stopwatch.

Suppose you are investigating Newton's second law by measuring the acceleration of a cart as it is being pulled by a falling weight of mass *m* (Figure **2**). (There are weights in the cart and one by one are moved from the cart and attached to the vertical string so that the mass of the system is constant.) Almost certainly there is a frictional force *f* between the cart and the table surface. If you neglect to take this force into account, you would expect the cart's acceleration *a* to be:

$$a = \frac{mg}{M}$$

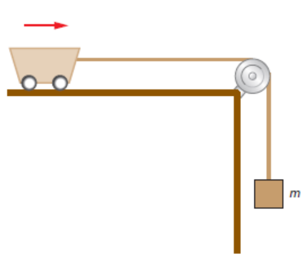


Figure 2 The falling block accelerates the cart.

where *M* is the constant combined mass of the system.

The graph of the acceleration versus *m* would be a straight line through the origin, as shown by the red line in Figure **3**. If you actually do the experiment, you will find that you do get a straight line, but not through the origin (blue line in Figure **3**). There is a negative intercept on the vertical axis.

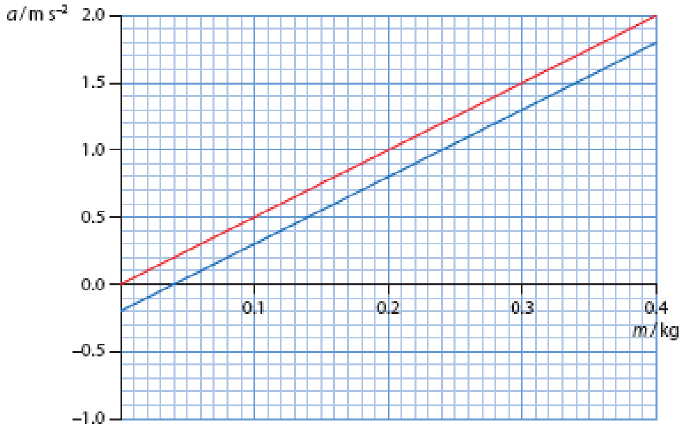


Figure 3 The variation of acceleration with falling mass with (blue) and without (red) frictional forces.

This is because with the frictional force present, Newton's second law predicts that:

$$a = \frac{mg}{M} - \frac{f}{M}$$

So a graph of acceleration *a* versus mass *m* would give a straight line with a negative intercept on the vertical axis.

Common systematic errors	
Zero error	Instrument does not show zero when it should.
Parallax error	Instrument viewed from wrong angle.
Calibration error	Instrument used gives different results from a standard reference instrument.
Reaction time	Instrument started too late.

Random uncertainties

The presence of **random uncertainty** is revealed when repeated measurements of the same quantity show a spread of values, some too large some too small. Unlike systematic uncertainties, which are always biased to be in the same direction, random uncertainties are unbiased. Suppose you ask ten people to use stopwatches to measure the time it takes an athlete to run a distance of 100 m. They stand by the finish line and start their stopwatches when the starting pistol fires. You will most likely get ten different values for the time. This is because some people will start/stop the stopwatches too early and some too late. You would expect that if you took an average of the ten times you would get a better **estimate** for the time than any of the individual measurements: the measurements fluctuate about some value. Averaging many measurements gives a more **accurate** estimate of the result. (See the section on accuracy and precision, overleaf.)

We include within random uncertainties, **reading** uncertainties (which really is a different type of uncertainty altogether). These have to do with the precision with which we can read an instrument. Suppose we use a ruler to record the position of the right end of an object, Figure **4**.

The first ruler has graduations separated by 0.2 cm. We are confident that the position of the right end is greater than 23.2 cm and smaller than 23.4 cm. The true value is somewhere between these bounds. The average of the lower and upper bounds is 23.3 cm and so we quote the measurement as $(23.3 \pm 0.1)\text{cm}.$ Notice that the uncertainty of ± 0.1 cm is **half the smallest graduation** on the ruler (0.2 cm).

Now let us use a ruler with a finer scale. We are again confident that the position of the right end is greater than 32.3 cm and smaller than 32.4 cm. The true value is somewhere between these bounds. The average of the bounds is 32.35 cm so we quote a measurement of $(32.35 \pm 0.05)\text{cm}.$ Notice again that the uncertainty of ± 0.05 cm is half the smallest graduation on the ruler (0.1 cm). This gives the general rule for analogue instruments:

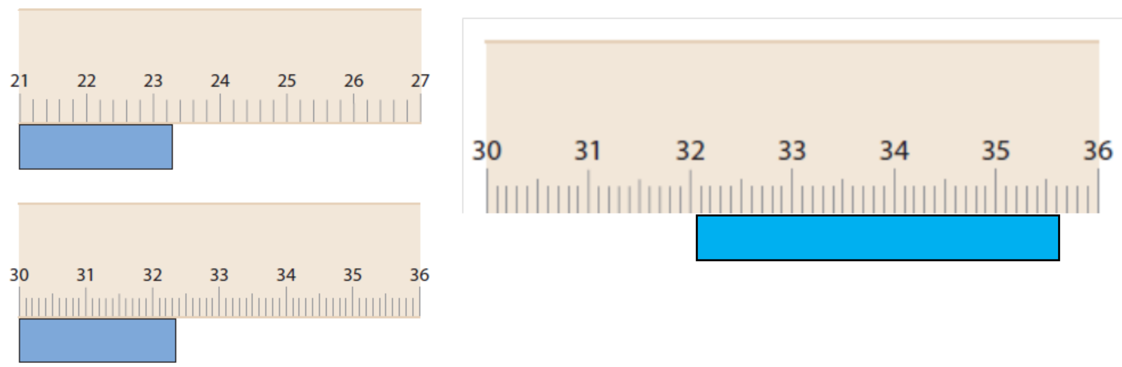


Figure 4 Two rulers with different graduations. The top has a width between graduations of 0.2 cm and the other 0.1 cm. To measure a length we need to make two measurements, one at each end.

The uncertainty in reading an instrument is ± half of the smallest width of the graduations on the instrument.

To measure a length of, say a rod, you must record the position of both ends of the rod. In Figure **4**, the left end is between 32.0 cm and 32.1 cm so we quote (32.05 ± 0.05) cm. The right end is between 35.6 cm and 35.7 so we quote (35.65 ± 0.05) cm. The length of the object is then (4.6 ± 0.1) cm. This is because the uncertainties at each end add up as we will explain in the next section.

For digital instruments, we may take the reading uncertainty to be the smallest division that the instrument can read. So, a stopwatch that reads time to two decimal places, e.g. 25.38 s, will have a reading uncertainty of ± 0.01 s, and a weighing scale that records a mass as 184.5 g will have a reading uncertainty of ± 0.1 g. Table **6** shows the typical precision for some common instruments.

Instrument	Precision of measurement
standard ruler	± 0.5 mm for each end of object so ± 1 mm
vernier calipers	± 0.1 mm
micrometer	± 0.01 mm
electronic weighing scale	± 0.1 g

stopwatch	± 0.01 s

Table 6 Typical precision for some common instruments.

Accuracy and precision

In physics, a measurement is said to be **accurate** if the systematic uncertainty in the measurement is small. This means in practice that the measured value is very close to the accepted value for that quantity (assuming that this is known – it is not always). The term **error** is used to indicate the difference between the measured value and the accepted value of that quantity: $e = Q_{\text{measured}} - Q_{\text{accepted}}$.

A measurement is said to be **precise** if the random uncertainty is small. This means in practice that when the measurement is repeated many times, the individual measurements are close to each other. We normally illustrate the concepts of accuracy and precision with the diagrams in Figure **5**: the red stars indicate individual measurements. The 'true' value is represented by the common centre of the three circles, the 'bull's-eye'. Measurements are precise if they are clustered together. They are accurate if they are close to the centre. The descriptions of three of the diagrams are obvious; the bottom right clearly shows results that are not precise because they are not clustered together. But they are accurate because their average value is roughly in the centre.

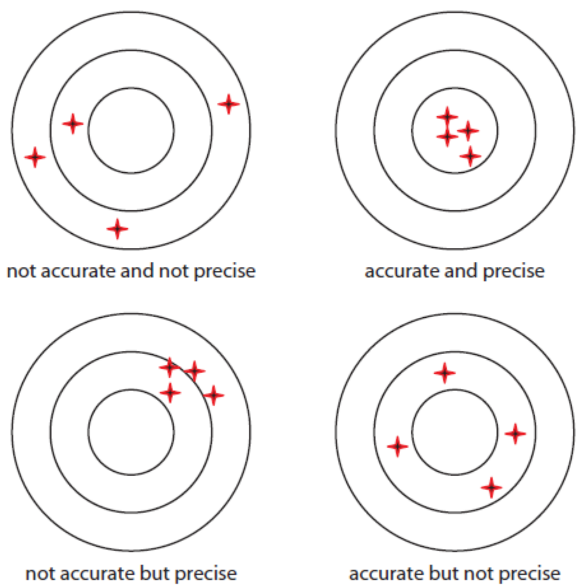
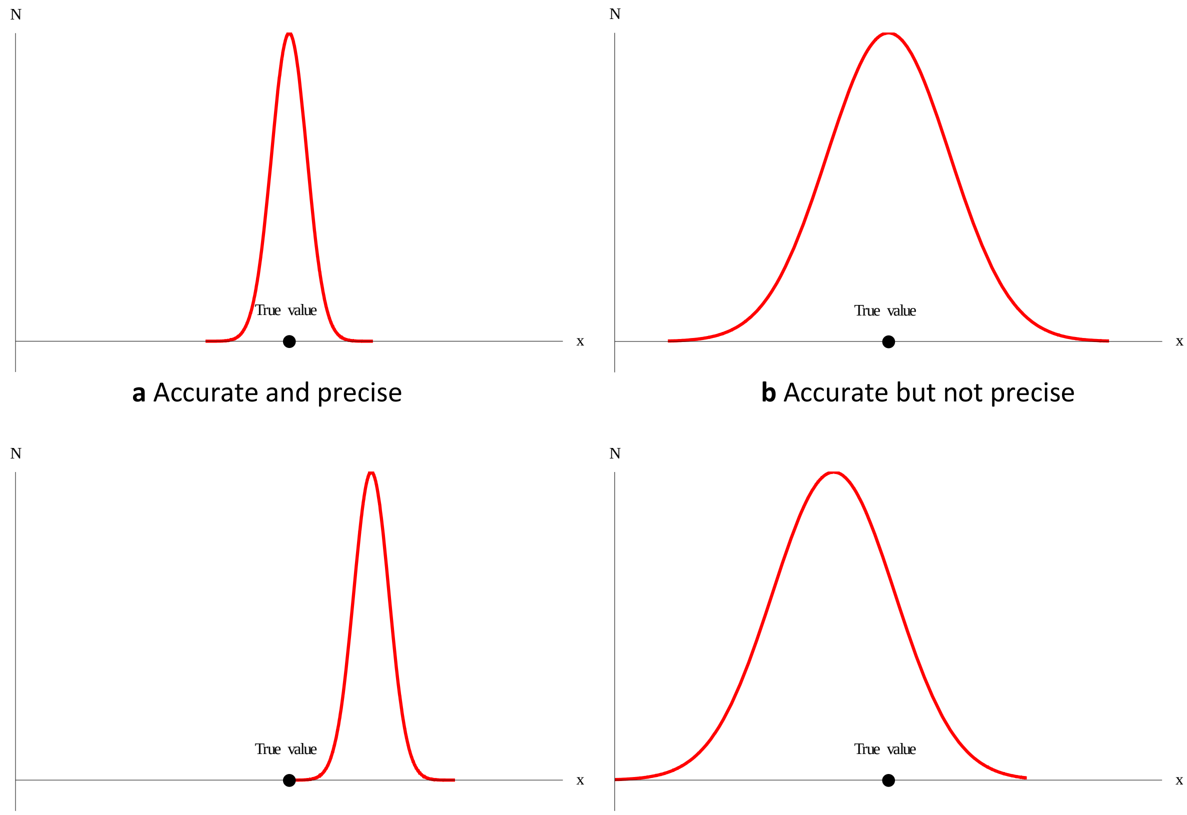


Figure 5 The meaning of accurate and precise measurements. Four different sets of four measurements each are shown.

The same idea is represented in Figure **6**: Suppose we measure a quantity *x very many* times and record the number *N* a particular value of *x* shows up. We will get a normal distribution. The following graphs show four possibilities.



a Accurate and precise

b Accurate but not precise

c Not accurate but precise

d Not accurate and not precise

Figure 6 The meaning of accurate and precise measurements.

In **a** the mean is very close to the actual true value of the quantity (accurate) and the spread of values is very small (precise). In **b** the mean is again close to the true value so the measurements are accurate but the spread of values is large (not precise). And similarly for **c** and **d**.

Averages

In an experiment a measurement must be repeated many times, if at all possible. If it is repeated N times and the results of the measurements are $x_1, x_2, \ldots, x_N$, we calculate the **mean** or the **average** of these values ($\bar{x}$) using:

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N}$$

This average is the **best estimate** for the quantity x based on the N measurements. What about the uncertainty? One way is to get the **standard deviation** of the N numbers using your calculator. Standard deviation will not be examined but you may need to use it for your Internal Assessment, so it is a good idea to learn it – you will learn it in your mathematics class anyway. The standard deviation σ of the N measurements is given by the formula (the calculator finds this very easily):

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_N - \bar{x})^2}{N}} \text{ or } \sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_N - \bar{x})^2}{N-1}}$$

(For our purposes, it makes little difference whether N or $N - 1$ is used in the denominator.)

A simpler method (not entirely satisfactory but acceptable for this course) is to use as an estimate of the uncertainty the quantity:

$$\Delta x = \frac{x_{max} - x_{min}}{2}$$

i.e. half of the difference between the largest and the smallest value.

You will often see uncertainties with 2 s.f. in the scientific literature. For example, the mass of the electron is quoted as: $m_e = (9.10938291 \pm 0.00000040) \times 10^{-31}$ kg. This is perfectly all right and reflects the experimenters' **level of confidence** in their results. Expressing the uncertainty to 2 s.f. implies a more sophisticated statistical analysis of the data than what is normally done in a high school physics course. With a lot of data, the measured values of the electron mass form a normal distribution with a mean $m_e = 9.109\ 382\ 91 \times 10^{-31}$ kg and standard deviation $0.00000040 \times 10^{-31}$ kg. The experimenter is then 68% confident that the true value of the mass lies within a standard deviation of the mean i.e. in the interval [9.109 382 51 × 10^{-31} kg, 9.109 383 31 × 10^{-31} kg].

Worked example

1.6 The diameter of a steel ball is to be measured using a micrometer. The following are sources of uncertainty:

1 The ball is not centred between the jaws of the micrometer.
2 The jaws of the micrometer are tightened too much.
3 The temperature of the ball may change during the measurement.
4 The ball may not be perfectly round.

Determine which of these are random and which are systematic sources of uncertainty.

Answer

Sources 3 and 4 lead to unpredictable results, so they are random uncertainties. Source 2 means that the measurement of diameter is always smaller since the micrometer is tightened too much, so this is a systematic source of uncertainty. Source 1 certainly leads to unpredictable results depending on how the ball is centred, so it is a random source of uncertainty. But since the ball is not centred the 'diameter' measured is always smaller than the true diameter, so this is also a source of systematic uncertainty.

Propagation of uncertainties

A measurement of a length may be quoted as L = (25.0 ± 0.5) cm. The value 25.0 is called the **best estimate** or the **mean value** of the measurement and the 0.5 cm is called the **absolute uncertainty** in the measurement. The ratio of absolute uncertainty to mean value is called the **fractional uncertainty**. Multiplying the fractional uncertainty by 100% gives the **percentage uncertainty**. So, for L = (25.0 ± 0.5) cm we have that:

- mean value = 25.0 cm
- absolute uncertainty = 0.5 cm
- fractional uncertainty $= \frac{0.5}{25.0} = 0.02$
- percentage uncertainty $= 0.02 \times 100\% = 2.0\%$

Notice that in this course we express uncertainties to 1 s.f. and the place value of the digit in the uncertainty must match the best estimate. Thus

Acceptable	Not acceptable
3.28 ± 0.02	5.52 ± 0.2 or 5.52 ± 0.002
23.4 ± 0.1	14.4 ± 1 or 14.4 ± 0.01
34.515 ± 0.003	28.115 ± 0.04 or 28.115 ± 0.0004
48 ± 2	56 ± 0.2

In general, if $a = a_0 \pm \Delta a$, we have:

- a_0 is the mean value of a
- absolute uncertainty = Δa
- fractional uncertainty $= \dfrac{\Delta a}{a_0}$
- percentage uncertainty $= \dfrac{\Delta a}{a_0} \times 100\%$

Suppose that three quantities are measured in an experiment: $a = a_0 \pm \Delta a$, $b = b_0 \pm \Delta b$, $c = c_0 \pm \Delta c$. We now wish to calculate a quantity Q in terms of a, b, c. For example, if a, b, c are the sides of a rectangular block we may want to find $Q = ab$, which is the area of the base, or $Q = 2a + 2b$, which is the perimeter of the base, or $Q = abc$, which is the volume of the block. Because of the uncertainties in a, b, c there will be an uncertainty in the calculated quantities as well. How do we calculate this uncertainty?

There are three cases to consider. We will give the results without proof.

Addition and subtraction

The first case involves the operations of addition and subtraction. For example, we might have $Q = a + b$ or $Q = a - b$ or $Q = a + b - c$. Then, in all cases the absolute uncertainty in Q is the **sum** of the **absolute** uncertainties in a, b and c.

$$Q = a + b \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b$$
$$Q = a - b \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b$$
$$Q = a + b - c \quad \Rightarrow \quad \Delta Q = \Delta a + \Delta b + \Delta c$$

Exam tip

In addition and subtraction, we always add the absolute uncertainties, never subtract.

If we multiply a quantity by a precisely known number, e.g.: $Q = 5a$ then $\Delta Q = 5\Delta a$ and if $Q = \pi a$ then $\Delta Q = \pi \Delta a$.

Worked examples

1.7 A ruler with 1 mm graduations is used to measure the side a of a square. One end of the square is at (2.00 ± 0.05) cm and the other end at (14.40 ± 0.05) cm side a of a square. Find the length of the side and the perimeter P of the square.

Answer

a = 14.40 – 2.00 = 12.40 cm. The uncertainty is 0.05 + 0.05 = 0.1 cm. Hence the side is (12.4 ± 0.1) cm.

Because $P = 4a$, the perimeter is 49.6 cm. The absolute uncertainty in P is: $\Delta P = 4\Delta a = 0.4$ cm.

Thus, $P = (49.6 \pm 0.4)\,\text{cm}$.

1.8 Find the percentage uncertainty in the quantity $Q = a - b$, where a = 538.7 ± 0.3 and b = 537.3 ± 0.5. Comment on the answer.

Answer

The calculated value is 1.4 and the absolute uncertainty is 0.3 + 0.5 = 0.8. So Q = 1.4 ± 0.8.

The fractional uncertainty is $\frac{0.8}{1.4} = 0.57$, so the percentage uncertainty is 57%.

The fractional uncertainty in the quantities a and b is quite small. But the numbers are close to each other so their difference is very small. This makes the fractional uncertainty in the difference unacceptably large.

Multiplication and division

The second case involves the operations of multiplication and division. Here the **fractional uncertainty** of the result is the sum of the fractional uncertainties of the quantities involved (k is a precisely known number):

$$Q = kab \Rightarrow \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0}$$

$$Q = k\frac{a}{b} \Rightarrow \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0}$$

(k does not appear in the expression for $\frac{\Delta Q}{Q_0}$)

$$Q = k\frac{ab}{c} \Rightarrow \frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0} + \frac{\Delta c}{c_0}$$

Powers and roots

The third case involves calculations where quantities are raised to powers or roots are taken. Here the fractional uncertainty of the result is the fractional uncertainty of the quantity multiplied by the absolute value of the power or the root (k is any precisely known number):

$$Q = ka^n \Rightarrow \frac{\Delta Q}{Q_0} = |n|\frac{\Delta a}{a_0}$$ (k does not appear in the expression for $\frac{\Delta Q}{Q_0}$)

$$Q = \sqrt[n]{a} \Rightarrow \frac{\Delta Q}{Q_0} = \frac{1}{n}\frac{\Delta a}{a_0}$$ (k does not appear in the expression for $\frac{\Delta Q}{Q_0}$)

Worked examples

1.9 (a)The sides of a rectangle are measured to be a = (4.4 ± 0.1) cm and b = (5.2 ± 0.1) cm. Find the area A of the rectangle.

(b)The radius of a sphere is R = (12.4 ± 0.1) cm. What is the volume of the sphere?

Answer

(a) The area A_0 is:

$A_0 = 4.4 \times 5.2 = 22.88\ \text{cm}^2$.

$$\frac{\Delta A}{22.88} = \frac{\Delta a}{a} + \frac{\Delta b}{b} = \frac{0.1}{4.4} + \frac{0.1}{5.2} = 0.041958$$

$\Rightarrow \Delta A = 22.88 \times 0.041958 = 0.96 \approx 1\text{cm}^2$ to 1 s.f. Hence A = (22.88 ± 1) cm^2 . To match the place value of the uncertainty: A = (23 ± 1) cm^2.

(b) $V = \frac{4\pi}{3}R^3$ so $V_0 = \frac{4\pi}{3} \times 12.4^3 = 7.986 \times 10^3\ \text{cm}^3$.

$\frac{\Delta V}{7.986 \times 10^3} = 3 \times \frac{\Delta R}{R} = 3 \times \frac{0.1}{12.4} = 2.419 \times 10^{-2}$. (The constant $\frac{4\pi}{3}$ does not appear in $\frac{\Delta V}{V_0}$.)

Hence $\Delta V = 7.986 \times 10^3 \times 2.419 \times 10^{-2} = 1.93 \times 10^2 \approx 2 \times 10^2\ \text{cm}^3 = 0.2 \times 10^3\ \text{cm}^3$ to 1 s.f.

Then, $V = (7.986 \pm 0.2) \times 10^3\ \text{cm}^3$. To match the place value of the uncertainty: $V = (8.0 \pm 0.2) \times 10^3\ \text{cm}^3$.

1.10 A mass is measured to be m = (4.4 ± 0.2) kg and its speed v is measured to be (18 ± 2) m s^{-1}. Find the kinetic energy of the mass.

Answer

The kinetic energy is $E = \frac{1}{2}mv^2$, so the mean value of the kinetic energy, E_0, is:

$$E_0 = \frac{1}{2} \times 4.4 \times 18^2 = 712.8\,\text{J}$$

Using:

$$\frac{\Delta E}{E_0} = \frac{\Delta m}{m_0} + \underbrace{2\times}_{\substack{\text{because of} \\ \text{the square}}} \frac{\Delta v}{v_0}$$

we find:

$$\frac{\Delta E}{712.8} = \frac{0.2}{4.4} + 2 \times \frac{2}{18} = 0.267677$$

So: $\Delta E = 712.8 \times 0.267677 = 190.8\,\text{J} \approx 2 \times 10^2\ \text{J}$ to 1 s.f. Thus, E = (7.13 ± 2) × 10^2 J. To match the place value of the uncertainty: E = (7 ± 2) × 10^2 J.

1.11 The length of a simple pendulum is increased by 4%. What is the fractional increase in the pendulum's period?

Answer

The period T is related to the length L through $T = 2\pi\sqrt{\frac{L}{g}}$.

Because this relationship has a square root, the fractional uncertainties are related by:

$$\frac{\Delta T}{T_0} = \underbrace{\frac{1}{2}}_{\text{because of the square root}} \times \frac{\Delta L}{L_0}$$ (the 2π does not appear here and neither does g)

We are told that $\frac{\Delta L}{L_0} = 4\%$. This means: $\frac{\Delta T}{T_0} = \frac{1}{2} \times 4\% = 2\%$.

1.12 A quantity Q is measured to be $Q = 34 \pm 5$. Calculate the uncertainty in **a** $\frac{1}{Q}$, **b** Q^2 and **c** $\sqrt{Q}$.

Answer

a $\frac{1}{Q} = Q^{-1}$, so

$$\frac{1}{Q} = \frac{1}{34} = 0.029412$$

$$\frac{\Delta(Q^{-1})}{0.029412} = |-1| \times \frac{\Delta Q}{Q} = \frac{5}{34} = 0.147$$

$$\Rightarrow \quad \Delta(Q^{-1}) = 0.029412 \times 0.147 = 0.0043\,25 \approx 0.004$$

Hence: $\frac{1}{Q} = 0.029 \pm 0.004$.

b $Q^2 = 34^2 = 1156$

$$\frac{\Delta(Q^2)}{1156} = 2 \times \frac{\Delta Q}{Q} = 2 \times \frac{5}{34} = 2 \times 0.147 = 0.29412$$

$$\Rightarrow \quad \Delta(Q^2) = 1156 \times 0.29412 = 340 \approx 3 \times 10^2 = 0.3 \times 10^3$$

Hence: $Q^2 = (1.2 \pm 0.3) \times 10^3$

c $\sqrt{Q} = \sqrt{34} = 5.83095$.

$$\frac{\Delta(\sqrt{Q})}{5.83095} = \frac{1}{2} \times \frac{\Delta Q}{Q} = \frac{1}{2} \times \frac{5}{34} = 0.073529 \Rightarrow \Delta(\sqrt{Q}) = 0.429.$$

Hence $\sqrt{34} = 5.8 \pm 0.4$.

1.13 The volume of a cylinder of base radius r and height h is given by $V = \pi r^2 h$. The volume is measured with an uncertainty of 4% and the height with an uncertainty of 2%. Determine the uncertainty in the radius.

Answer

We must first solve for the radius to get $r = \sqrt{\frac{V}{\pi h}}$. The percentage uncertainty is then:

$$\frac{\Delta r}{r}\times 100\% = \frac{1}{2}\left(\frac{\Delta V}{V}+\frac{\Delta h}{h}\right)\times 100\% = \frac{1}{2}(4\%+2\%) = 3\%$$

Putting it all together

In an experiment you have to measure a certain quantity. If possible you will measure it very many times, say *N*. You will use the average of your measurements as the measured value of that quantity. What is the uncertainty that will be quoted for the average value?

Suppose that each measurement x_i is subject to the same uncertainty Δx. The uncertainty in $x_1+\cdots+x_N$ is (with the rules we learned in the previous section) $N\Delta x$ and so the uncertainty in the average is $\frac{N\Delta x}{N}=\Delta x$. So Δx is one possibility for the uncertainty in the average. A second possibility is the quantity $\frac{x_{\max}-x_{\min}}{2}$. A third is the standard deviation of the measurements. (But it is unlikely that you will measure a quantity more than 3 or say 5 times. This is a small number and the standard deviation in this case does not make much sense.) Which one do we use? In this course, the conservative approach is to take the largest of the three.

As we just saw, making many measurements of the same quantity does not reduce the uncertainty; but we get a more accurate result *if there are no systematic errors*; values above and below the average tend to cancel out. The average is a better estimate of the quantity that is being measured compared to any one individual measurement. If systematic errors are present, averaging many measurements offers no improvement to accuracy.

Suppose we measure the period of a pendulum (in seconds) ten times with a stopwatch that measures to the nearest 0.01 s:

1.20, 1.25, 1.30, 1.13, 1.25, 1.17, 1.41, 1.32, 1.29, 1.30 (all in seconds)

We calculate the mean:

$$\bar{t} = \frac{t_1+t_2+\ldots+t_{10}}{10} = 1.2620\,\text{s}$$

and

$$\Delta t = \frac{t_{\max}-t_{\min}}{2} = \frac{1.41-1.13}{2} = 0.140\,\text{s}$$

and the standard deviation: $\sigma = 0.081$ s.

But each measurement is subject to our reaction time. If this is 0.1 s then starting and stopping the stopwatch generates an uncertainty of 0.2 s which is way larger than the precision of 0.01 s of the stopwatch.

So we have to choose between $\Delta t = 0.2\,\text{s}$, 0.081 s and 0.14 s .

We choose the largest so here we have Δt = 0.2 s. The uncertainty is in the first decimal place. **The value of the average period must also be expressed to the same precision as the uncertainty**, i.e. to one decimal place here, $\bar{t} = 1.3\,\text{s}$. We then state that: period = $(1.3 \pm 0.2)\,\text{s}$.

There is a percentage uncertainty of 15% which is too high. We must re-evaluate our methodology; measuring a single period was not a good idea. We should have measured 10 periods and divided by 10. The time for 10 periods would be subject to an uncertainty of 0.2 s but dividing by 10 to get the period reduces the uncertainty to 0.2 s.

As far as your experiments, your IA and your extended essay are concerned, all you have to do is provide any reasonable treatment of uncertainties and you will never be penalized for doing something that is not rigorously sound mathematically.

Best-fit lines

In mathematics, plotting a point on a set of axes is straightforward. In physics, it is slightly more involved because the point consists of measured or calculated values and so is subject to uncertainty. So the point $\left(x_0 \pm \Delta x, y_0 \pm \Delta y\right)$ is plotted as shown in Figure **7**. The uncertainties are represented by **uncertainty bars**. To 'go through the uncertainty bars' a best-fit line can go through the shaded area.

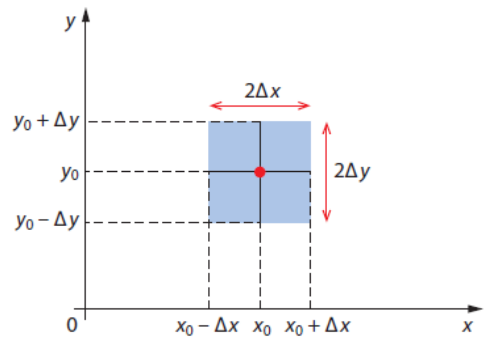


Figure 7 A point plotted along with its uncertainty bars.

In a physics experiment we usually try to plot quantities that will give straight-line graphs. The graph in Figure **8** shows the variation with extension *x* of the tension *T* in a spring. The points and their uncertainty bars are plotted. The blue line is the best-fit line. It has been drawn by eye by trying to minimise the distance of the points from the line – this means that some points are above and some are below the best-fit line.

The gradient (slope) of the best-fit line is found by using two points **on the best-fit line** as far from each other as possible. We use (0, 0) and (0.0390, 7.88). The gradient is then:

$$\text{gradient} = \frac{\Delta F}{\Delta x}$$

$$\text{gradient} = \frac{7.88 - 0}{0.0390 - 0}$$

$$\text{gradient} = 202 \text{ N m}^{-1}$$

The best-fit line has equation $F = 202x$. (The vertical intercept is essentially zero; in this equation *x* is in metres and *F* in newtons.)

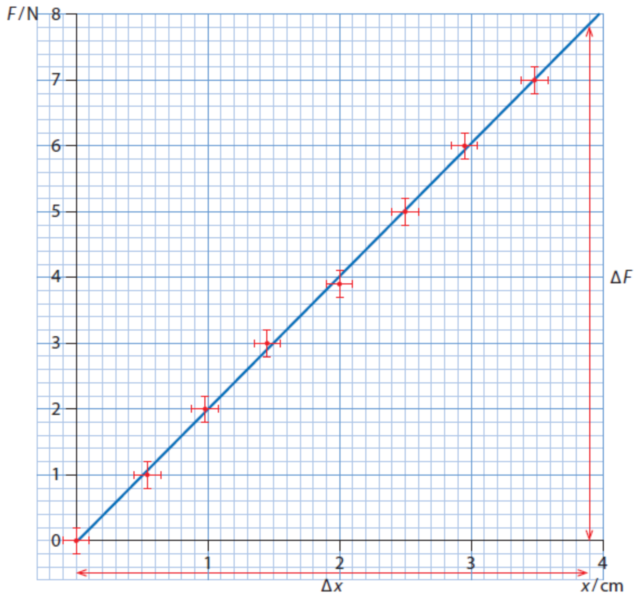


Figure 8 Data points plotted together with uncertainties in the values for the tension. To find the gradient, use two points on the best-fit line far apart from each other.

Uncertainties in the gradient and intercept

When the best-fit line is a straight line we can easily obtain uncertainties in the gradient and the vertical intercept. The idea is to draw lines of maximum and minimum gradient in such a way that they go through all the error bars (not just the ‘first’ and the ‘last’ points). Figure **9** shows the best-fit line (in blue) and the lines of maximum and minimum gradient. The green line is the line through all uncertainty bars of greatest gradient. The red line is the line through all uncertainty bars with smallest gradient. All lines are drawn by eye.

The green line has gradient k_{max} = 210 N m^{-1} and intercept −0.18 N. The red line has gradient k_{min} = 193 N m^{-1} and intercept +0.13 N. So we can find the uncertainty in the gradient as:

$$\Delta k = \frac{k_{max} - k_{min}}{2} = \frac{210 - 193}{2} = 8.5 \approx 8 \text{N} \text{m}^{-1}$$

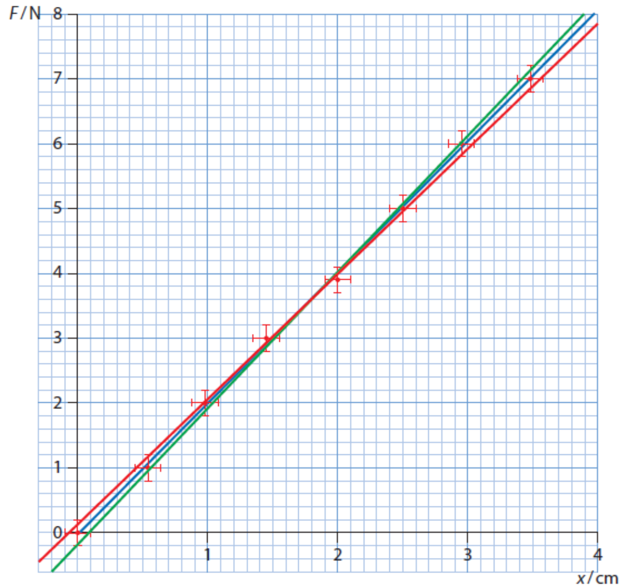


Figure 9 The best-fit line, along with lines of maximum and minimum gradient.

The uncertainty in the vertical intercept is similarly:

$$\Delta_{\text{intercept}} = \frac{0.13-(-0.18)}{2} = 0.155 \approx 0.2\text{N}$$

We saw earlier that the line of best fit has gradient 202 N m^{-1} and zero intercept. So we quote the results as k = (2.02 ± 0.08) ×10^2 N m^{-1} and vertical intercept = 0.0 ± 0.2 N.

Nature of science

A key part of the scientific method is recognising the uncertainties that are present in the experimental technique being used, and working to reduce these as much as possible. In this section you have learned how to calculate uncertainties in quantities that are combined in different ways and how to estimate uncertainties from graphs. You have also learned how to recognise systematic and random uncertainties.

No matter how much care is taken, scientists know that their results are uncertain. But they need to distinguish between inaccuracy and uncertainty, and to know how confident they can be about the validity of their results. The search to gain more accurate results pushes scientists to try new ideas and refine their techniques. There is always the possibility that a new result may confirm a hypothesis for the present, or it may overturn current theory and open a new area of research. Being aware of doubt and uncertainty are key to driving science forward.

Test yourself

23 The magnitudes of two forces are measured to be (120 ± 5) N and (60 ± 3) N. Find the sum and difference of the two magnitudes, giving the uncertainty in each case.

24 The quantity Q depends on the measured values a and b in the following ways:

a $Q = \frac{a}{b}, a = 20 \pm 1, b = 10 \pm 1$

b $Q = 2a + 3b$, $a = 20 \pm 2$, $b = 15 \pm 3$

c $Q = a - 2b$, $a = 50 \pm 1$, $b = 24 \pm 1$

d $Q = a^2$, $a = 10.0 \pm 0.3$

e $Q=\frac{a^2}{b^2}, a=100\pm 5, b=20\pm 2$

In each case, find the value of *Q* and its absolute and percentage uncertainty.

25 The centripetal force is given by $F=\frac{mv^2}{r}$. The mass is measured to be (2.8 ± 0.1) kg, the velocity (14 ± 2) m s^{-1} and the radius (8.0 ± 0.2) m; find the force on the mass, including the uncertainty.

26 The radius *r* of a circle is measured to be (2.4 ± 0.1) cm. Find the uncertainty in:

a the area of the circle

b the circumference of the circle.

27 The sides of a rectangle are measured as (4.4 ± 0.2) cm and (8.5 ± 0.3) cm. Find the area and perimeter of the rectangle.

28 The length *L* of a pendulum is increased by 2%. Find the percentage increase in the period *T*.

$$\left(T=2\pi\sqrt{\frac{L}{g}}\right)$$

29 The volume of a cone of base radius *R* and height *h* is given by $V=\frac{\pi R^2 h}{3}$. The uncertainty in the radius and in the height is 4%. Find the percentage uncertainty in the volume.

30 In an experiment to measure current and voltage across a device, the following data were collected: (*V*, *I*) = {(0.1, 26), (0.2, 48), (0.3, 65), (0.4, 90)}.The current was measured in mA and the voltage in mV. The uncertainty in the current was ± 4 mA. Plot the current versus the voltage and draw the best-fit line through the points. Suggest whether the current is proportional to the voltage.

31 In a similar experiment to that in question **30**, the following data were collected for current and voltage: (*V*, *I*) = {(0.1, 27), (0.2, 44), (0.3, 60), (0.4, 78)} with an uncertainty of ± 4 mA in the current. Plot the current versus the voltage and draw the best-fit line. Suggest whether the current is proportional to the voltage.

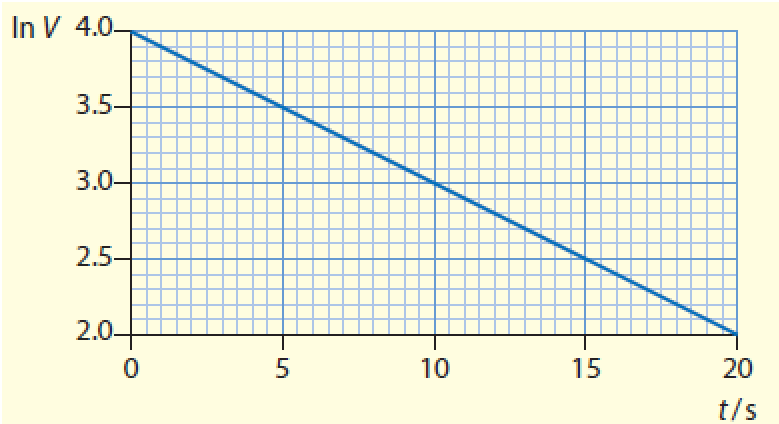
32 A circle and a square have the same perimeter. Which shape has the larger area?

33 The graph shows the natural logarithm of the voltage across a capacitor of capacitance *C* = 5.0 μF as a function of time. The voltage is given by the equation $V = V_0 e^{-t/RC}$, where *R* is the resistance of the circuit. Find:

a the initial voltage

b the time for the voltage to be reduced to half its initial value

c the resistance of the circuit.



34 The table shows the mass M of several stars and their corresponding luminosity L (power emitted).

a Plot L against M and draw the best-fit line.

b Plot the logarithm of L against the logarithm of M. Use your graph to find the relationship between these quantities, assuming a power law of the kind $L = kM^{\alpha}$. Give the numerical value of the parameter α.

Mass M (in solar masses)	Luminosity L (in terms of the Sun's luminosity)
1.0 ± 0.1	1 ± 0
3.0 ± 0.3	42 ± 4
5.0 ± 0.5	230 ± 20
12 ± 1	4700 ± 50
20 ± 2	26 500 ± 300

1.3 Vectors and scalars

The physical quantities we will meet in this course are either scalars (i.e. they just have magnitude) or vectors (i.e. they have magnitude and direction). This section provides the tools you need for dealing with vectors.

Vectors

Some quantities in physics, such as time, distance, mass, speed and temperature, just need one number to specify them. These are called **scalar** quantities. For example, it is sufficient to say that the mass of a body is 64 kg or that the temperature is −5.0 °C. On the other hand, many quantities are fully specified only if, in addition to a number, a direction is needed. Saying that you will leave Paris now, in a train moving at 220 km/h, does not tell us where you will be in 30 minutes because we do not know the **direction** in which you will travel. Quantities that need a direction in addition to magnitude are called **vector** quantities. Table **7** gives some examples of vectors and scalars.

Vectors	Scalars
displacement	distance
velocity	speed
acceleration	mass
force	time
weight	density
electric field	electric potential
magnetic field	electric charge
gravitational field	gravitational potential
momentum	temperature
area	volume
angular velocity	work/energy/power

Table 7 Examples of vectors and scalars.

A vector is represented by a straight arrow, as shown in Figure **10**. The direction of the arrow represents the direction of the vector and the length of the arrow represents the **magnitude** of the vector. To say that two vectors are the same means that **both** magnitude and direction are the same. The vectors in Figure **10a** are all equal to each other. In other words, vectors do not have to start from the same point to be equal. In Figure **10b** the two vectors have the same magnitude but opposite direction.

We write vectors as italic boldface $\boldsymbol{a}$ or $\vec{a}$. The magnitude is written as $|\boldsymbol{a}|$, or $|\vec{a}|$ or just a.

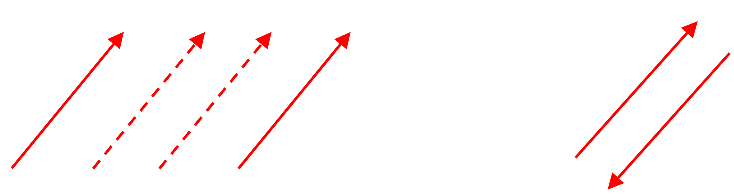


Figure 10 a Representation of vectors by arrows. A vector shifted parallel to itself results in the same vector. **b** These vectors have the same magnitude but opposite direction.

Multiplication of a vector by a number

A vector can be multiplied by a number. The vector $\boldsymbol{a}$ multiplied by the positive number 2 gives a vector in the same direction as $\boldsymbol{a}$ but 2 times longer. The vector $\boldsymbol{a}$ multiplied by the negative number −0.5 is opposite to $\boldsymbol{a}$ in direction and half as long (Figure **11**). The vector $-\boldsymbol{a}$ has the same magnitude as $\boldsymbol{a}$ but is opposite in direction.

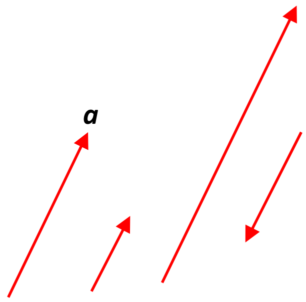


Figure11 Multiplication of a vector ***a*** by a number results in vectors that either parallel or anti-parallel to ***a***.

Addition of vectors

There are two ways to add two vectors together.

Figure **12** shows a blue vector $\vec{b}$ and a red vector $\vec{r}$. We want to find the vector $\vec{b}+\vec{r}$, the sum of $\vec{b}$ and $\vec{r}$. The sum is the single vector that has the same effect as $\vec{b}$ and $\vec{r}$ together.

Figure **12** shows the two methods used in adding two vectors.

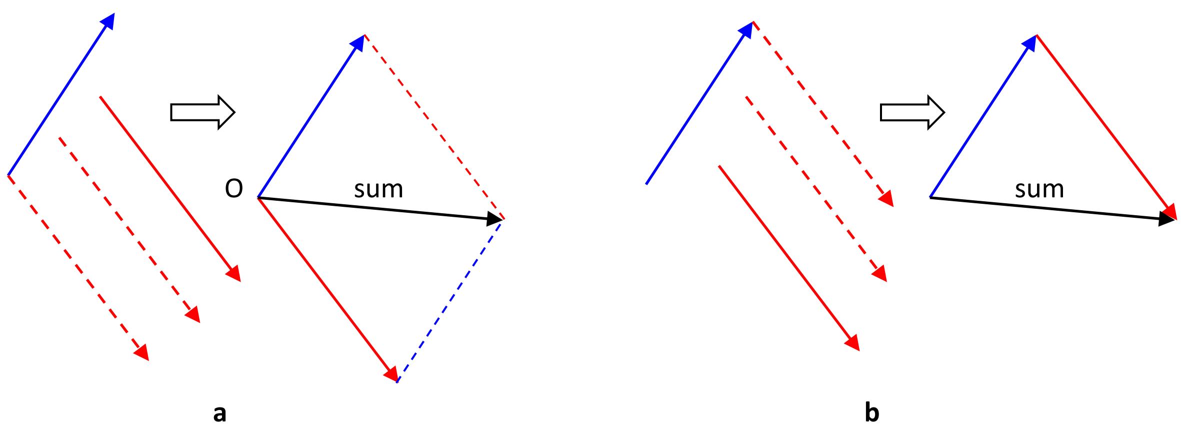


Figure 12 Vectors $\vec{b}$ and $\vec{r}$ are to be added. **a** The parallelogram method. **b** The triangle method.

To add two vectors with the parallelogram method:

1 Shift them so they start at a common point O.

2 Complete the parallelogram whose sides are $\vec{b}$ and $\vec{r}$.

3 Draw the diagonal of this parallelogram starting at O. This is the vector $\vec{b}+\vec{r}$.

To add two vectors with the triangle method:

1 Shift one vector (the red, say) so that it begins where the other (blue) ends.

2 Join the beginning of blue to the end of red. This is the vector $\vec{b}+\vec{r}$.

The second method is used when more than two vectors need to be added, Figure **13**.

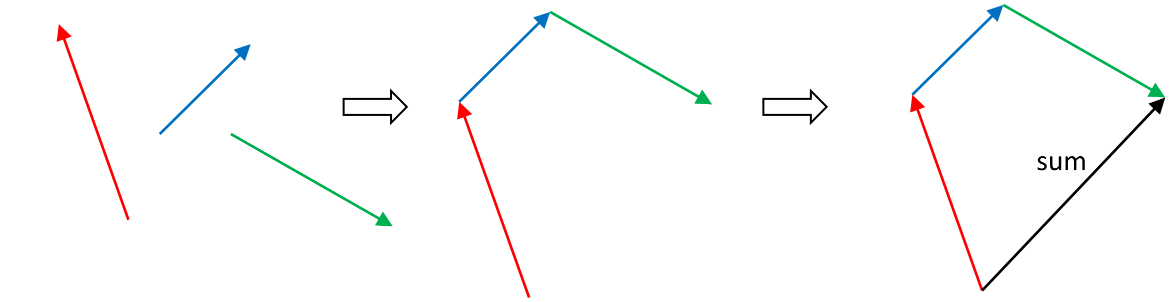


Figure 13 Adding more than two vectors.

Exam tip

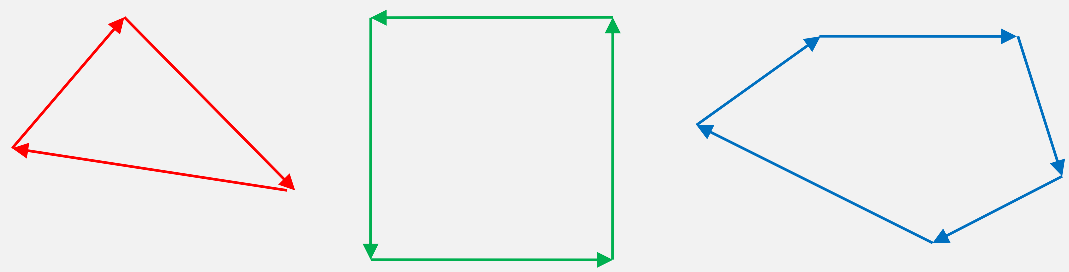


Figure 14
Vectors (with arrows pointing in the same sense) forming closed polygons add up to zero.

Subtraction of vectors

Figure **15** shows two vectors $\vec{b}$ and $\vec{r}$. We want to find the vector that equals $\vec{b} - \vec{r}$.

To subtract two vectors:

1 Reverse the direction of the vector that is being subtracted, (here $\vec{r}$).

2 Now simply follow the rule for addition to **add** $\vec{b}$ to $(-\vec{r})$.

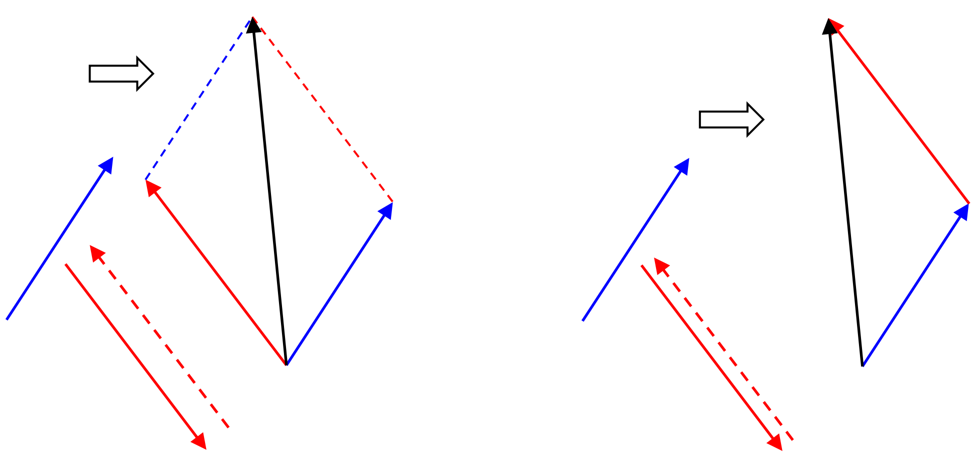


Figure 15 Subtraction of vectors. Here we find $\vec{b} - \vec{r}$.

Figure **16** shows the subtraction $\vec{r} - \vec{b}$.

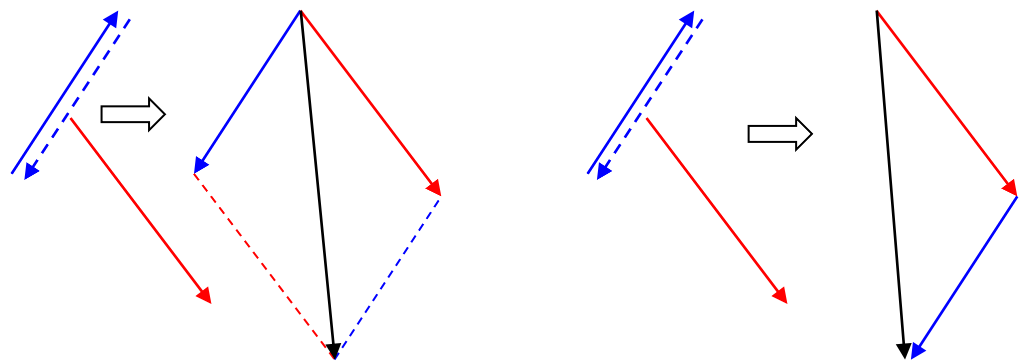


Figure 16 Subtraction of vectors: $\vec{r} - \vec{b}$.

The result is a vector that has the same magnitude but opposite direction to the vector $\vec{b} - \vec{r}$.

Worked examples

1.14 Copy the diagram in Figure **17a**. Use the diagram to draw the third force that will keep the point P in equilibrium.

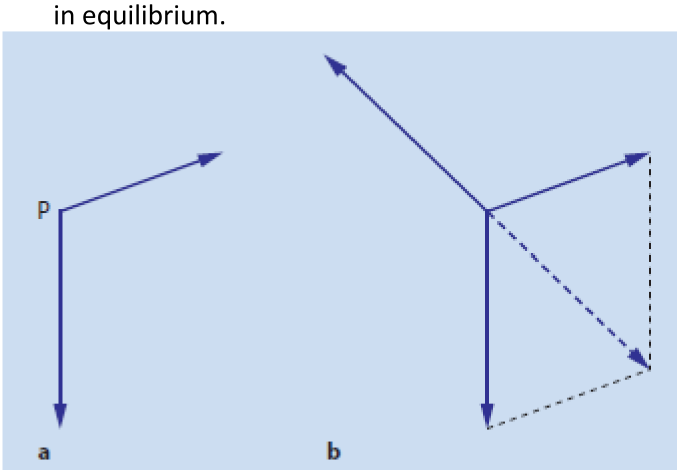


Figure 17

Answer

We find the sum of the two given forces using the parallelogram rule and then draw the opposite of that vector, as shown in Figure **17b**.

1.15 A velocity vector of magnitude 1.2 m s^{-1} is horizontal. A second velocity vector of magnitude 2.0 m s^{-1} must be added to the first so that the sum is vertical in direction. Find the direction of the second vector and the magnitude of the sum of the two vectors.

Answer

We need to draw a scale diagram, as shown in Figure **18**. Representing 1.0 m s^{-1} by 2.0 cm, we see that the 1.2 m s^{-1} corresponds to 2.4 cm and 2.0 m s^{-1} to 4.0 cm.
First draw the horizontal vector. Then mark the vertical direction from O. Using a compass (or a ruler),

mark a distance of 4.0 cm from A, which intersects the vertical line at B. AB must be one of the sides of the parallelogram we are looking for.
Now measure a distance of 2.4 cm horizontally from B to C and join O to C. This is the direction in which the second velocity vector must be pointing. Measuring the diagonal OB (i.e. the vector representing the sum), we find 3.2 cm, which represents 1.6 m s^{-1}. Using a protractor, we find that the 2.0 m s^{-1} velocity vector makes an angle of about 37° with the vertical.

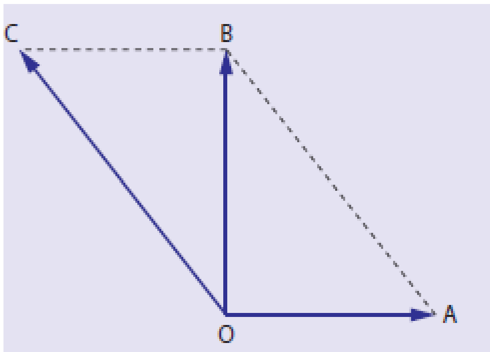


Figure 18 Using a scale diagram to solve a vector problem.

1.16 A person walks 5.0 km east, followed by 3.0 km north and then another 4.0 km east. Find their final position.

Answer

The walk consists of three steps. We may represent each one by a vector (Figure **19**).

- The first step is a vector of magnitude 5.0 km directed east (**OA**).
- The second is a vector of magnitude 3.0 km directed north (**AB**).
- The last step is represented by a vector of 4.0 km directed east (**BC**).

The person will end up at a place that is given by the vector sum of these three vectors, that is **OA** + **AB** + **BC**, which equals the vector **OC**. By measurement from a scale drawing, or by simple geometry, the distance from O to C is 9.5 km and the angle to the horizontal is 18.4°.

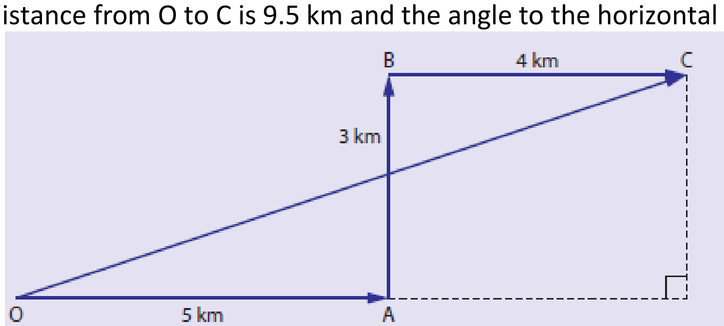


Figure 19 Scale drawing using 1 cm = 1 km.

Vectors corresponding to line segments are shown as bold capital letters, for example **OA**. The magnitude of the vector is the length OA and the direction is from O towards A.

1.17 A body moves in a circle of radius 3.0 m with a constant speed of 6.0 m s^{-1}. The velocity vector is at all times tangent to the circle. The body starts at A, proceeds to B and then to C. Find the change in the velocity vector between A and B and between B and C (Figure **20**).

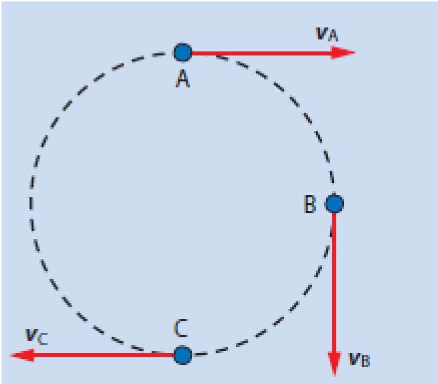


Figure 20

Answer

For the velocity change from A to B we have to find the difference $\boldsymbol{v}_B - \boldsymbol{v}_A$. and for the velocity change from B to C we need to find $\boldsymbol{v}_C - \boldsymbol{v}_B$. The vectors are shown in Figure **21**.

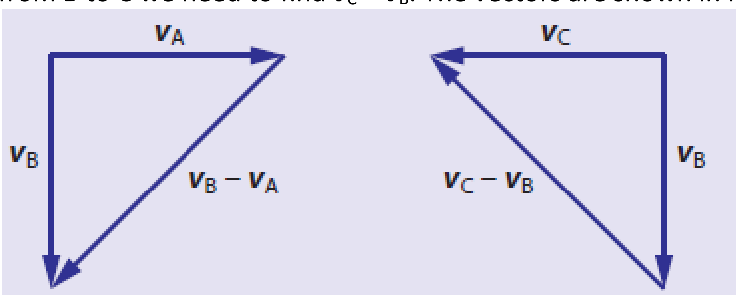


Figure 21

The vector $\boldsymbol{v}_B - \boldsymbol{v}_A$ is directed south-west and its magnitude is (by the Pythagorean theorem):

$$\begin{aligned}\sqrt{v_A{}^2 + v_B{}^2} &= \sqrt{6^2 + 6^2} \\ &= \sqrt{72} \\ &= 8.49\,\mathrm{ms}^{-1}\end{aligned}$$

The vector $\boldsymbol{v}_C - \boldsymbol{v}_B$ has the same magnitude as $\boldsymbol{v}_B - \boldsymbol{v}_A$ but is directed north-west.

Components of a vector

Suppose that we use perpendicular axes x and y and draw vectors on this x–y plane. We take the origin of the axes as the starting point of the vectors. (Other vectors whose beginning points are not at the origin can be shifted parallel to themselves until they, too, begin at the origin.) Given a vector $\boldsymbol{a}$ we define its **components along the axes** as follows. From the tip of the vector draw lines parallel to the axes and mark the point on each axis where the lines intersect the axes (Figure **22**). As seen in Figure **22**, formally, the components have a positive or negative sign depending on which side of the axis they fall on.

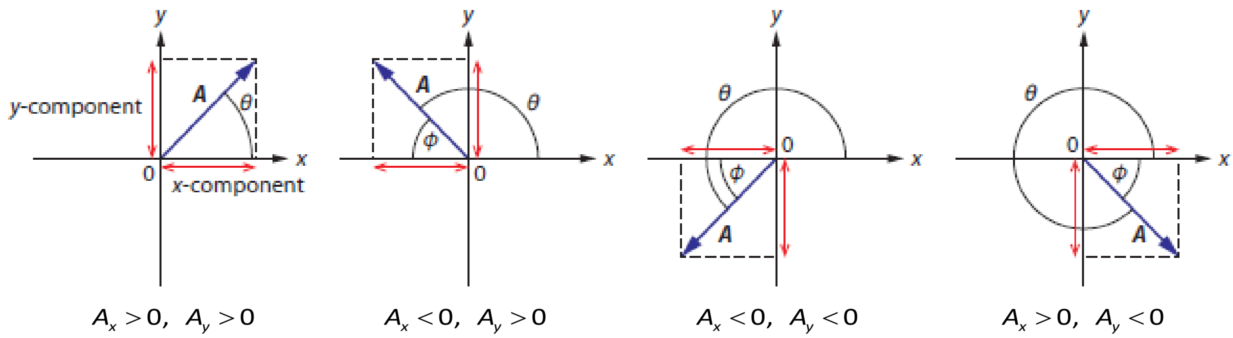


Figure 22 The components of a vector ***A*** and the angle needed to calculate the components. The angle θ is measured counter-clockwise from the positive *x*-axis.

The *x*- and *y*-components of ***A*** are called A_x and A_y. They are given by:

$A_x = A \cos \theta$
$A_y = A \sin \theta$

where *A* is the magnitude of the vector and θ **is the counter-clockwise angle between the vector and the positive** *x*-axis. This is the formula in your data booklet. But remember that in these formulas the angle θ **must** be the one defined in Figure **22**. But this angle is not always the most convenient, especially if it is greater than 90°. A more convenient angle to work with is the angle φ of Figure **22** but when using this angle the signs have to be put in by hand. This is shown in Worked example **1.18**.

Worked examples

1.18 Find the components of the vectors in Figure **23**. The magnitude of ***a*** is 12.0 units and that of ***b*** is 24.0 units.

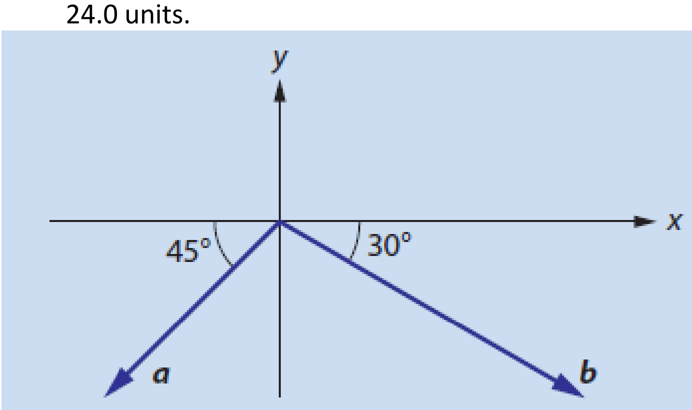


Figure 23

Answer

Taking the angle from the positive *x*-axis, the angle for ***a*** is $\theta = 180° + 45° = 225°$ and that for ***b*** is $\theta = 360° - 30° = 330°$. Thus:

$a_x = 12.0 \cos 225° = -8.49$ $\qquad b_x = 24.0 \cos 330° = 20.8$
$a_y = 12.0 \sin 225° = -8.49$ $\qquad b_y = 24.0 \sin 330° = -12.0$

But we do not have to use the awkward angles of 225° and 330°. For vector ***a*** it is better to use the angle of $\varphi = 45°$. In that case simple trigonometry gives:

$a_x = -12.0 \cos 45° = -8.49$ and $a_y = -12.0 \sin 45° = -8.49$

↑ put in by hand ↑ put in by hand

For vector ***b*** it is convenient to use the angle of $\varphi = 30°$, which is the angle the vector makes with the *x*-axis. But in this case:

$b_x = 24.0 \cos 30° = 20.8$ and $b_y = -24.0 \sin 30° = -12.0$

↑ put in by hand

1.19 Find the components of the vector ***W*** along the axes shown in Figure **24**.

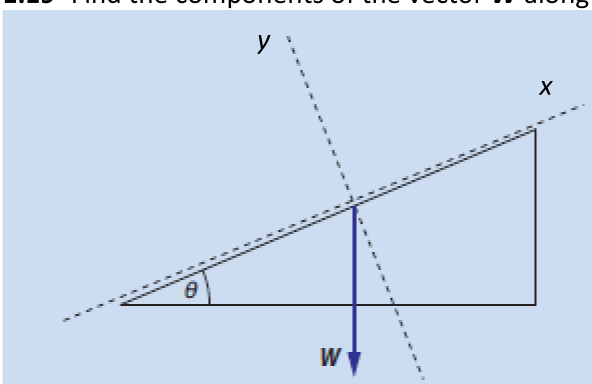


Figure 24.

Answer

See Figure **25**. Notice that the angle between the vector ***W*** and the negative *y*-axis is θ.
Then by simple trigonometry

$W_x = -W \sin \theta$ (W_x is opposite the angle θ so the sine is used)
$W_y = -W \cos \theta$ (W_y is adjacent to the angle θ so the cosine is used)

(Both components are along the negative axes, so a minus sign has been put in by hand.)

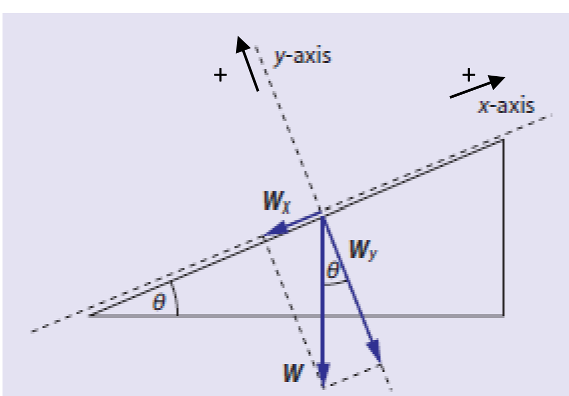


Figure 25

Reconstructing a vector from its components

Knowing the components of a vector allows us to reconstruct it (i.e. to find the magnitude and direction of the vector). Suppose that we are given that the *x*- and *y*-components of a vector are F_x and F_y. We need to find the magnitude of the vector F and the angle θ it makes with the *x*-axis (Figure **26**). The magnitude is found by using the Pythagorean theorem and the angle by using the definition of tangent.

$$F = \sqrt{F_x^{\,2} + F_y^{\,2}}, \qquad \theta = \arctan\frac{F_y}{F_x}$$

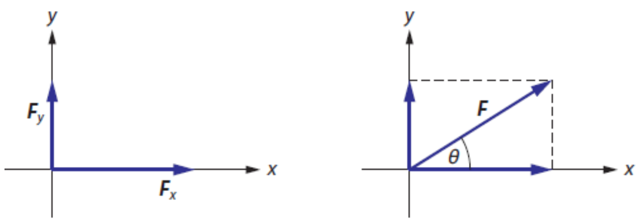


Figure 26 Given the components of a vector we can find its magnitude and direction.

In general, the simplest procedure to find the angle without getting stuck in trigonometry is to decide which quadrant the vector lies in and then evaluate $\varphi = \arctan\left|\frac{F_y}{F_x}\right|$ i.e. **ignore the signs** in the components. The calculator will then give you the angle between the vector and the *x*-axis, as shown in Figure **27**.

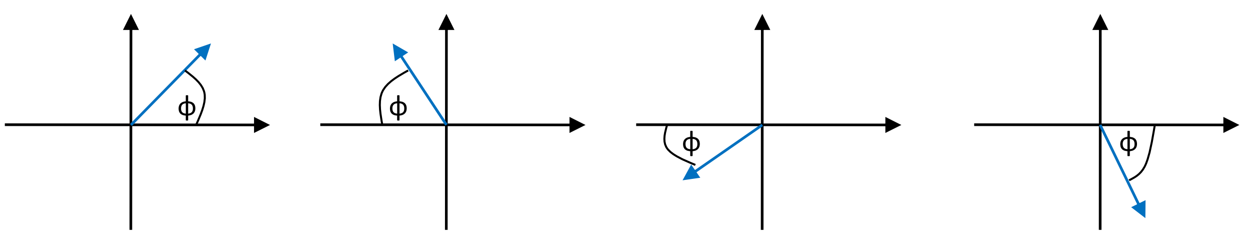


Figure 27 The angle $\varphi = \arctan\left|\frac{F_y}{F_x}\right|$ is the angle shown in each diagram.

So, for example, the vector with components $F_x = -12$ and $F_y = 5.0$ is in the second quadrant, Figure **28**, and its direction is given by $\varphi = \arctan\left|\frac{F_y}{F_x}\right| = \arctan\frac{5.0}{12} = 22.6^\circ \approx 23^\circ$.

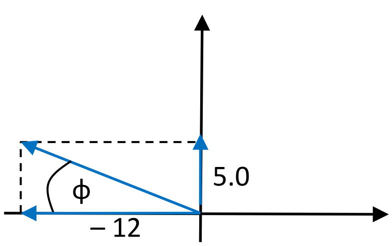


Figure 28 The angle φ giving the direction of the vector.

Here is another example. We need to find the magnitude and direction of the vector with components $F_x = 2.0$ and $F_y = -4.0$. The vector lies in the fourth quadrant, as shown in Figure **29**.

The magnitude is:

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(2.0)^2 + (-4.0)^2} = \sqrt{20} = 4.47 \approx 4.5\,.$$

The direction is found from:

$$\theta = \arctan\left|\frac{F_y}{F_x}\right| = \arctan\left|\frac{-4}{2}\right| = \arctan 2 \approx 63^\circ\,.$$

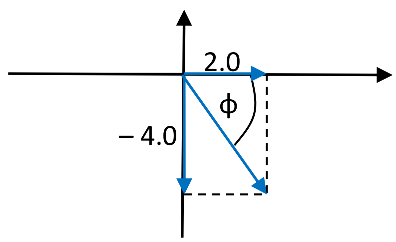


Figure 29 The vector is in the third quadrant.

This angle is the one shown in Figure **29**.

Adding or subtracting vectors is very easy when we have the components, as Worked example **1.20** shows.

Worked example

1.20 Find the sum of the vectors shown in Figure **30**. $\boldsymbol{F}_1$ has magnitude 8.0 units and $\boldsymbol{F}_2$ has magnitude 12 units. Their directions are as shown in the diagram.

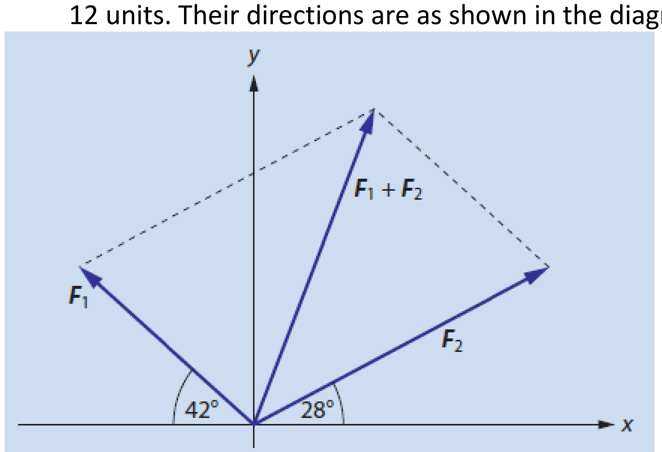


Figure 30 The sum of vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ (not to scale).

Answer

Find the components of the two vectors:

$F_{1x} = -F_1 \cos 42° = -5.945$ $\qquad$ $F_{1y} = F_1 \sin 42° = 5.353$

$F_{2x} = F_2 \cos 28° = 10.595$ $\qquad$ $F_{2y} = F_2 \sin 28° = 5.634$

The sum $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$ then has components: $F_x = F_{1x} + F_{2x} = 4.650$ and $F_y = F_{1y} + F_{2y} = 10.987$.

The magnitude of the sum is therefore $F = \sqrt{4.650^2 + 10.987^2} = 11.9 \approx 12$.

and its direction is (the vector is in the first quadrant): $\varphi = \arctan\left(\frac{10.987}{4.65}\right) = 67.1 \approx 67°$.

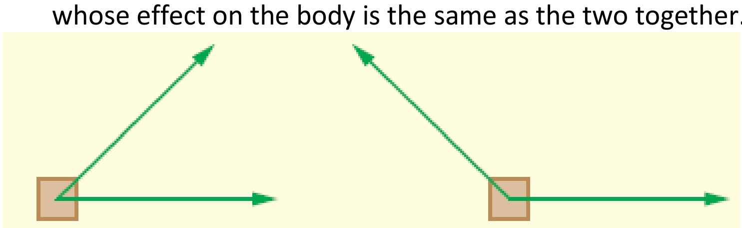
Nature of science

For thousands of years, people across the world have used maps to navigate from one place to another, making use of the ideas of distance and direction to show the relative positions of places. The concept of vectors and the algebra used to manipulate them were introduced in the first half of the 19th century to represent real and complex numbers in a geometrical way. Scientists and mathematicians saw that this model could be applied to theoretical physics, and by the middle of the 19th century vectors were being used to model problems in electricity and magnetism.

Resolving a vector into components and reconstructing the vector from its components are useful mathematical techniques for dealing with measurements in three-dimensional space. These mathematical techniques are invaluable when dealing with physical quantities that have both magnitude and direction, such as calculating the effect of multiple forces on an object. In this section we have done this in two dimensions, but vector algebra can be applied to three dimensions and more.

Test yourself

35 A body is acted upon by the two forces shown in the diagram. In each case draw the one force whose effect on the body is the same as the two together.



36 Vector ***A*** has a magnitude of 12.0 units and makes an angle of 30° with the positive *x*-axis. Vector ***B*** has a magnitude of 8.00 units and makes an angle of 80° with the positive *x*-axis. Using a graphical method, find the magnitude and direction of the vectors:

a $\boldsymbol{A} + \boldsymbol{B}$ **b** $\boldsymbol{A} - \boldsymbol{B}$ **c** $\boldsymbol{A} - 2\boldsymbol{B}$

37 Repeat the previous problem, this time using components.

38 Find the magnitude and direction of the vectors with components:

a $A_x = -4.0$ cm, $A_y = -4.0$ cm

b $A_x = 124$ km, $A_y = -158$ km

c $A_x = 0$, $A_y = -5.0$ m

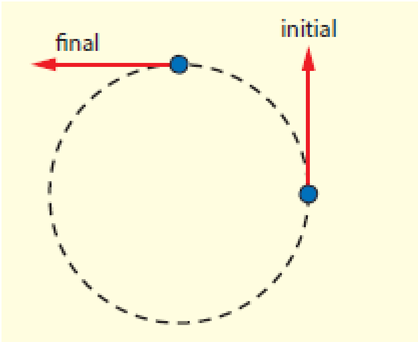
d $A_x = 8.0$ N, $A_y = 0$

39 The components of vectors ***A*** and ***B*** are as follows: ($\boldsymbol{A}_x = 2.00$, $\boldsymbol{A}_y = 3.00$), ($\boldsymbol{B}_x = -2.00$, $\boldsymbol{B}_y = 5.00$). Find the magnitude and direction of the vectors:

a $\boldsymbol{A}$ **b** $\boldsymbol{B}$ **c** $\boldsymbol{A} + \boldsymbol{B}$ **d** $\boldsymbol{A} - \boldsymbol{B}$ **e** $2\boldsymbol{A} - \boldsymbol{B}$

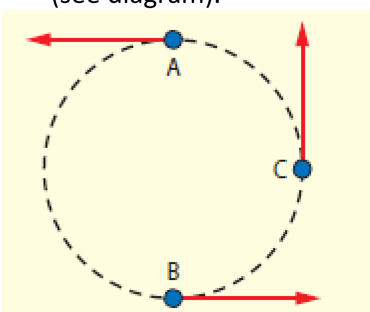
40 The position vector of a moving object has components ($r_x = 2$, $r_y = 2$) initially. After a certain time the position vector has components ($r_x = 5$, $r_y = 6$). Find the displacement vector.

41 The diagram shows the velocity vector of a particle moving in a circle with speed 10 m s^{-1} at two separate points. The velocity vector is tangential to the circle. Find the vector representing the **change** in the velocity vector.



42 In a certain collision, the momentum vector of a particle changes direction but not magnitude. Let ***p*** be the momentum vector of a particle suffering an elastic collision and changing direction by 30°. Find, in terms of p (= |***p***|), the magnitude of the vector representing the change in the momentum vector.

43 The velocity vector of an object moving on a circular path has a direction that is tangent to the path (see diagram).



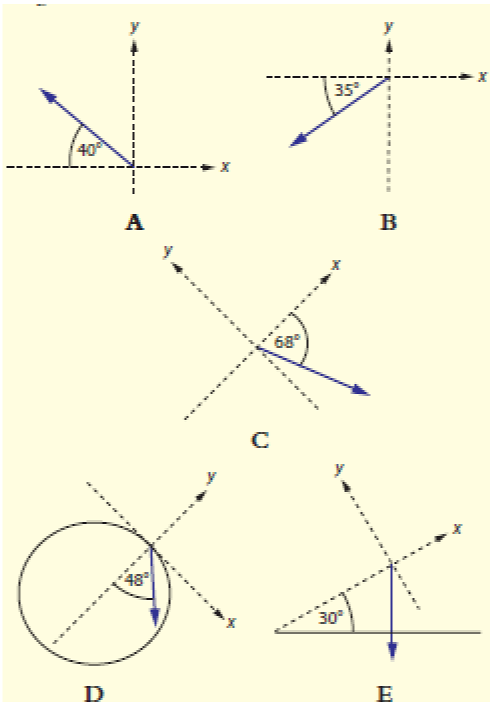
If the speed (magnitude of velocity) is constant at 4.0 m s^{-1}, find the change in the velocity vector as the object moves:

a from A to B

b from B to C.

c What is the change in the velocity vector from A to C? How is this related to your answers to **a** and **b**?

44 For each diagram, find the components of the vectors along the axes shown. Take the magnitude of each vector to be 10.0 units.



45 Vector **A** has a magnitude of 6.00 units and is directed at 60° to the positive *x*-axis. Vector **B** has a magnitude of 6.00 units and is directed at 120° to the positive *x*-axis. Find the magnitude and direction of vector **C** such that ***A*** + ***B*** + ***C*** = 0. Place the three vectors so that one begins where the previous ends. What do you observe?

46 Plot the following pairs of vectors on a set of *x*- and *y*-axes. The angles given are measured counter-clockwise from the positive *x*-axis. Then, using the algebraic component method, find their sum in magnitude and direction.

a 12.0 N at 20° and 14.0 N at 50°

b 15.0 N at 15° and 18.0 N at 105°

c 20.0 N at 40° and 15.0 N at 310° (i.e. −50°)

47 Two vectors have magnitudes 5 and 9 units. What is the smallest and largest possible magnitude of the sum of the two vectors.

48 A ball is attached to a string that makes an angle of 60° to the vertical. What are the components of the tension force *T* and the weight *W* along the two sets of axes shown?